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THE UNIVERSAL DEFINING EQUATIONS OF ABELIAN SURFACES WITH LEVEL 3 STRUCTURE

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1. Preliminaries

Let $A=\mathbb{C}^2/\tau\mathbb{Z}^2+\mathbb{Z}^2$, $\tau\in\mathbb{H}_2$ be a principally polarized abelian surface, and we put $\Lambda=\Lambda_1\oplus\Lambda_2=\tau\mathbb{Z}^2+\mathbb{Z}^2$. Let H be a hermitian form on $\mathbb{C}^2$ given by $(\operatorname{Im}\tau)^{-1}$, and $E=\operatorname{Im}H$, that is, $E(v,w)=\operatorname{Im}(v(\operatorname{Im}\tau)^{-1}\overline{w})={}^tv_1w_2-{}^tv_2w_1$ for $v=\tau v_1+v_2$ and $w=\tau w_1+w_2$. In particular (1): $E(\Lambda,\Lambda)\subset\mathbb{Z}$. We define $\alpha:\Lambda\to\mathbb{C}^\times$ by $\alpha(\lambda)=(-1)^{{}^t\lambda_1\lambda_2}$ for $\lambda=\tau\lambda_1+\lambda_2$. Then (2): $\alpha(\lambda+\mu)=\alpha(\lambda)\alpha(\mu)\exp\pi iE(\lambda,\mu)$.

We put $L_0=L(H,\alpha)$, that is the quotient of trivial line bundle $\mathbb{C}\times\mathbb{C}^2$ by the action of Λ

$$((a,v),\lambda)\longmapsto(e_\lambda(v)a,v+\lambda)\quad a\in\mathbb{C},v\in\mathbb{C}^2,\lambda\in\Lambda$$

$$e_\lambda(v)=\alpha(\lambda)\exp(\pi H(v,\lambda)+\frac{\pi}{2}H(\lambda,\lambda)).$$

Let $L=L_0^k=L(kH,\alpha^k)$, $K(L)=\{x\in A\mid T_x^*L\cong L\}=A_k=\{x\in A\mid kx=0\}$. We decompose $K(L)=K(L)_1\oplus K(L)_2$. Then the Riemann-Roch theorem says

$$\dim H^0(A,L)=\sharp K(L)_1=\sharp K(L)_2.$$

We can construct the standard basis of $H^0(A,L)$. Let B be a symmetric form on $\mathbb{C}^2$ given by $B(v,w)={}^tv(\operatorname{Im}\tau)^{-1}w$. We consider $M=L(H,\alpha)$ for any pair (H,α) which satisfies (1) and (2). For $x\in K(M)_1$, we define

$$\vartheta_x^M(v)=\exp\left(\frac{\pi}{2}B(v,v)-\frac{\pi}{2}(H-B)(x+2v,x)\right)\sum_{\lambda\in\Lambda_1}\exp\left(\pi(H-B)(x+v,\lambda)-\frac{\pi}{2}(H-B)(\lambda,\lambda)\right),$$

then $\{\vartheta_x^M\}_{x\in K(M)_1}$ form a basis of $H^0(A,M)$.

Now we put $k=3$, $L=L_0^3$. Then the canonical map

$$\varphi:\bigoplus_{n=0}^\infty\operatorname{Sym}^nH^0(A,L)\longrightarrow\bigoplus_{k=0}^\infty H^0(A,L^n)$$

is surjective by the theorem of Koizumi([Ko1, Corollary 4.7]), and $\ker\varphi$ is generated by the elements of degree 2 and 3 by the theorem of Sekiguchi([S, Main Theorem]).

2. Quadratic equations

We consider the map $\varphi_2:\operatorname{Sym}^2H^0(A,L)\to H^0(A,L^2)$. Then $\dim\ker\varphi_2=45-36=9$, thus we have 9 linearly independent equations.

Lemma 1 (Addition formula). *We denote $Z_2=K(L^2)_1\cap A_2$. For any $x_1,x_2\in K(L)_1$,*

$$\varphi_2(\vartheta_{x_1}^L\vartheta_{x_2}^L)=\sum_{z\in Z_2}\vartheta_{y_2+z}^{L^2}(0)\cdot\vartheta_{y+z}^{L^2}$$

with $y_1, y_2 \in K(L^2)_1$ *such that* $y + y_2 = x_1$ *and* $y - y_2 = x_2$.

For the proof, see [LB, (1.3), Chapter 7], or also see [M1, p339].

Now we use the following notation: $K(L^2)_1 = \{{}^t(a,b) \mid a,b \in \{0,3,6,9,12,15\}\}$, $K(L)_1 = \{{}^t(a,b) \mid a,b \in \{0,6,12\}\}$ and $Z_2 = \{{}^t(a,b) \mid a,b \in \{0,9\}\}$. Since $K(L^2)_1 = K(L)_1 \oplus Z_2$, we can take $K(L)_1$ as the representative system of $K(L^2)_1/Z_2$.

For $y \in K(L)_1$, let $W_y \subset H^0(A, L^2)$ be the space spanned by $\{\vartheta^{L^2}_{y+z}\}_{z\in Z_2}$ and $V_y \subset \operatorname{Sym}^2 H^0(A,L)$ be the space spanned by $\{\vartheta^L_{y+u}\vartheta^L_{y-u}\}_{u\in K(L)_1}$. Then φ_2 maps V_y onto W_y, and we write $\varphi_2^y : V_y \to W_y$ the restriction of φ_2. We decompose

$$\operatorname{Sym}^2 H^0(A,L) = \bigoplus_{y\in K(L)_1} V_y, \quad H^0(A,L^2) = \bigoplus_{y\in K(L)_1} W_y.$$

For simplicity, we write $X_{a,b} = \vartheta^L_y$, $Y_{a,b} = \vartheta^{L^2}_x$ and $q(y) = \vartheta^{L^2}_y(0)$, with y (or x) $= \binom{a}{b}$. Then φ_2^y is given by

$$\varphi_2^y(X_{y+u}X_{y-u}) = \sum_{z\in Z_2} q(u+z)Y_{y+z}.$$

For example the case of $y = {}^t(0,0) = 0$, we have

$$\varphi_2^0 \begin{pmatrix} X_{0,0}^2 \\ X_{6,0}X_{12,0} \\ X_{0,6}X_{0,12} \\ X_{6,6}X_{12,12} \\ X_{6,12}X_{12,6} \end{pmatrix} = \begin{pmatrix} q\binom{0}{0} & q\binom{0}{9} & q\binom{9}{0} & q\binom{9}{9} \\ q\binom{6}{0} & q\binom{6}{9} & q\binom{15}{0} & q\binom{15}{9} \\ q\binom{0}{6} & q\binom{0}{15} & q\binom{9}{6} & q\binom{9}{15} \\ q\binom{6}{6} & q\binom{6}{15} & q\binom{15}{6} & q\binom{15}{15} \\ q\binom{6}{12} & q\binom{6}{3} & q\binom{15}{12} & q\binom{15}{3} \end{pmatrix} \begin{pmatrix} Y_{0,0} \\ Y_{0,9} \\ Y_{9,0} \\ Y_{9,9} \end{pmatrix}.$$

We write M the 5×4 representation matrix above, and let M_k be the matrix removing the k-th row vector from M. Then for $h_k = (-1)^{k+1}\det M_k$, we see

$$(h_1, \ldots, h_5) \cdot {}^t(X_{0,0}^2, \ldots, X_{6,12}X_{12,6}) = 0.$$

Theorem 1. *We have* 9 *quadratic equations:*

$$\begin{aligned}
&(q1) && h_1X_{0,0}^2 + h_2X_{6,0}X_{12,0} + h_3X_{0,6}X_{0,12} + h_4X_{6,6}X_{12,12} + h_5X_{6,12}X_{12,6} = 0,\\
&(q2) && h_1X_{6,0}^2 + h_2X_{12,0}X_{0,0} + h_3X_{6,6}X_{6,12} + h_4X_{12,6}X_{0,12} + h_5X_{12,12}X_{0,6} = 0,\\
&(q3) && h_1X_{12,0}^2 + h_2X_{0,0}X_{6,0} + h_3X_{12,6}X_{12,12} + h_4X_{0,6}X_{6,12} + h_5X_{0,12}X_{6,6} = 0,\\
&(q4) && h_1X_{0,6}^2 + h_2X_{6,6}X_{12,6} + h_3X_{0,12}X_{0,0} + h_4X_{6,12}X_{12,0} + h_5X_{6,0}X_{12,12} = 0,\\
&(q5) && h_1X_{6,6}^2 + h_2X_{12,6}X_{0,6} + h_3X_{6,12}X_{6,0} + h_4X_{12,12}X_{0,0} + h_5X_{12,0}X_{0,12} = 0,\\
&(q6) && h_1X_{12,6}^2 + h_2X_{0,6}X_{6,6} + h_3X_{12,12}X_{12,0} + h_4X_{0,12}X_{6,0} + h_5X_{0,0}X_{6,12} = 0,\\
&(q7) && h_1X_{0,12}^2 + h_2X_{6,12}X_{12,12} + h_3X_{0,0}X_{0,6} + h_4X_{6,0}X_{12,6} + h_5X_{6,6}X_{12,0} = 0,\\
&(q8) && h_1X_{6,12}^2 + h_2X_{12,12}X_{0,12} + h_3X_{6,0}X_{6,6} + h_4X_{12,0}X_{0,6} + h_5X_{12,6}X_{0,0} = 0,\\
&(q9) && h_1X_{12,12}^2 + h_2X_{0,12}X_{6,12} + h_3X_{12,0}X_{12,6} + h_4X_{0,0}X_{6,6} + h_5X_{0,6}X_{6,0} = 0.
\end{aligned}$$

Here $h_k = (-1)^{k+1}\det M_k$ $(1 \le k \le 5)$.

We can regard q or h as functions in τ. Then

$$q\begin{pmatrix} 3a \\ 3b \end{pmatrix} = \sum_{m\in\mathbb{Z}^2} \exp 6\pi i\tau[m - \frac{1}{12}\begin{pmatrix} 2a \\ 2b \end{pmatrix}],$$

and each h_i is quartic polynomial in these q's, and belongs to $M_2(\Gamma^2(12))$.

Theorem 2. *For each $1 \leq k \leq 5$, h_k is contained in the space $M_2(\Gamma^2(3), \varepsilon)$, with a character ε of $\Gamma^2(3)/\Gamma^2(12)$ such that $\varepsilon^2 = 1$.*

For the proof of this theorem, we use the fact that the group $G = \Gamma^2(3)/\Gamma^2(12) \cong \Gamma^2/\Gamma^2(4)$, and G is generated by the elements

$$\begin{pmatrix} 1_2 & 3S \\ 0 & 1_2 \end{pmatrix}, \quad \begin{pmatrix} 1_2 & 0 \\ 3S & 1_2 \end{pmatrix}, \quad {}^tS = S.$$

The ring structure of the graded ring of Siegel modular forms of degree 2, level 3 is already known by Freitag and Salvati Manni [FS]. They showed

$$\bigoplus_{k=0}^{\infty} M_k(\Gamma^2(3)) = \mathbb{C}[t_1, \dots, t_5, f_1, \dots, f_5]$$

with $t_1, \dots, t_5 \in M_1(\Gamma^2(3))$ and $f_1, \dots, f_5 \in M_3(\Gamma^2(3))$. They have 5 relations in weight 5, and 15 relations in weight 6.

From this fact, we can rewrite the above functions $h_1, \dots, h_5$ by using $t_1, \dots, t_5$ and $f_1, \dots, f_5$ as follows.

$$\begin{aligned}
h_1^2 &= \frac{1}{216}(f_1 t_1 - t_1^4 + 4t_1(t_2^3 + t_3^3 + t_4^3 + t_5^3) - 24 t_2 t_3 t_4 t_5), \\
h_1 h_2 &= \frac{1}{108}(f_2 t_2 + 3t_1^2 t_2^2 - 12 t_1 t_3 t_4 t_5), \\
h_1 h_3 &= \frac{1}{108}(f_3 t_3 + 3t_1^2 t_3^2 - 12 t_1 t_2 t_4 t_5), \\
h_1 h_4 &= \frac{1}{108}(f_4 t_4 + 3t_1^2 t_4^2 - 12 t_1 t_2 t_3 t_5), \\
h_1 h_5 &= \frac{1}{108}(f_5 t_5 + 3t_1^2 t_5^2 - 12 t_1 t_2 t_3 t_4).
\end{aligned}$$

3. Cubic equations

Next we consider the map $\varphi_3 : \mathrm{Sym}^3 H^0(A, L) \to H^0(A, L^2)$. We need $\dim \ker \varphi_3 = 165 - 81 = 84$ relations. However, in this case, all the generators of $\ker \varphi_3$ is given by the theorem of Birkenhake and Lange.

Let $Z_6 = A_6 \cap K(L^2)_1$. For $\rho \in \widehat{Z}_6 = \mathrm{Hom}(Z_6, \mathbb{C}_1^\times)$, $y_1 \in K(L^6)_1$ and $y_2 \in K(L^2)_1$ we define

$$\theta_{(y_1, y_2), \rho}(v) = \sum_{a \in Z_6} \rho(a) \vartheta^{L^6}_{y_1 - a}(v) \vartheta^{L^2}_{y_2 - 3a}(v).$$

Theorem 3 (Cubic theta relations [BL, Theorem 3.3]). *Let L be an ample line bundle on A and assume $L = L_0^3$ for a line bundle L_0. Then all the cubic theta relations are given by the following form:*

$$\begin{aligned}
&\theta_{(y_1, y_2), \rho}(0) \sum_{b \in Z_6} \rho(b) \vartheta^L_{y_1' + y_2' + y_3 + 2b} \vartheta^L_{y_1' - y_2' + y_3 + 2b} \vartheta^L_{-2y_1' + y_3 + 2b} \\
&= \theta_{(y_1', y_2'), \rho}(0) \sum_{b \in Z_6} \rho(b) \vartheta^L_{y_1 + y_2 + y_3 + 2b} \vartheta^L_{y_1 - y_2 + y_3 + 2b} \vartheta^L_{-2y_1 + y_3 + 2b}.
\end{aligned}$$

Here $\rho \in \widehat{Z}_6$, $y_1, y_1' \in K(L^6)_1$, $y_2, y_2' \in K(L^2)_1$ and $y_3 \in K(L^3)_1$ such that

$$\begin{cases} y_1 + y_2 + y_3, \quad y_1 - y_2 + y_3, \quad -2y_1 + y_3, \\ y_1' + y_2' + y_3, \quad y_1' - y_2' + y_3, \quad -2y_1' + y_3 \end{cases}$$

belong to $K(L)_1$.

Using this theorem, we can write down all the 84 generators of $\ker\varphi_3$. Let $W_3 = \{0,3,6\}$, and $\widehat{Z}_6^+$ be the set of all the character ρ of Z_6 such that $\rho^3 \equiv 1$, that is, all the character of $W_3^2 \bmod 9 \cong (\mathbb{Z}/3\mathbb{Z})^2$. We define the character $\rho_1,\dots,\rho_4 \in \widehat{Z}_6^+$ by

$$\begin{cases}\rho_1\left(\begin{smallmatrix}3\\0\end{smallmatrix}\right)=1,\\ \rho_1\left(\begin{smallmatrix}0\\3\end{smallmatrix}\right)=\omega.\end{cases}\qquad \begin{cases}\rho_2\left(\begin{smallmatrix}3\\0\end{smallmatrix}\right)=\omega,\\ \rho_2\left(\begin{smallmatrix}0\\3\end{smallmatrix}\right)=1.\end{cases}\qquad \begin{cases}\rho_3\left(\begin{smallmatrix}3\\0\end{smallmatrix}\right)=\omega^2,\\ \rho_3\left(\begin{smallmatrix}0\\3\end{smallmatrix}\right)=\omega.\end{cases}\qquad \begin{cases}\rho_4\left(\begin{smallmatrix}3\\0\end{smallmatrix}\right)=\omega,\\ \rho_4\left(\begin{smallmatrix}0\\3\end{smallmatrix}\right)=\omega.\end{cases}$$

For $\rho \in \widehat{Z}_6^+$, we define

$$\theta^\rho\begin{pmatrix}x & y\\ z & w\end{pmatrix} = \sum_{a,b\in\mathbb{Z}/6\mathbb{Z}} \rho\begin{pmatrix}3a\\3b\end{pmatrix}\Theta\begin{bmatrix}6a-2x & 6b-2z\\ 18a-2y & 18b-2w\end{bmatrix},$$

with

$$\Theta\begin{bmatrix}a & b\\ c & d\end{bmatrix}(\tau) := \sum_{N\in M_2(\mathbb{Z})} \exp \pi i(\begin{pmatrix}18 & 0\\ 0 & 6\end{pmatrix}\left[N+\frac{1}{36}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right]\tau).$$

Theorem 4. *The following list contains all of the* 84 *linearly independent relations of degree* 3.

$$\text{(c1)}\qquad \sum_{(a,b)\in K(L)_1} X_{a,b}^3 = 3\,\frac{\theta^1\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^1\left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right)}(X_{0,0}X_{6,0}X_{12,0}+X_{0,6}X_{6,6}X_{12,6}+X_{0,12}X_{6,12}X_{12,12})$$

$$\text{(c2)}\qquad = 3\,\frac{\theta^1\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^1\left(\begin{smallmatrix}0&0\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{0,6}X_{0,12}+X_{6,0}X_{6,6}X_{6,12}+X_{12,0}X_{12,6}X_{12,12})$$

$$\text{(c3)}\qquad = 3\,\frac{\theta^1\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^1\left(\begin{smallmatrix}0&6\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{6,6}X_{12,12}+X_{0,6}X_{6,12}X_{12,0}+X_{0,12}X_{12,6}X_{6,0})$$

$$\text{(c4)}\qquad = 3\,\frac{\theta^1\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^1\left(\begin{smallmatrix}0&6\\0&12\end{smallmatrix}\right)}(X_{0,0}X_{6,12}X_{12,6}+X_{6,6}X_{0,12}X_{12,0}+X_{12,12}X_{0,6}X_{6,0})$$

$$\text{(d1)}\quad X_{0,0}^3+X_{6,0}^3+X_{12,0}^3-X_{0,6}^3-X_{6,6}^3-X_{12,6}^3 = 3\,\frac{\theta^{\rho_1}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_1}\left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right)}(X_{0,0}X_{6,0}X_{12,0}-X_{0,6}X_{6,6}X_{12,6}),$$

$$\text{(d2)}\quad X_{0,0}^3+X_{6,0}^3+X_{12,0}^3-X_{0,12}^3-X_{6,12}^3-X_{12,12}^3 = 3\,\frac{\theta^{\rho_1}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_1}\left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right)}(X_{0,0}X_{6,0}X_{12,0}-X_{0,12}X_{6,12}X_{12,12}),$$

$$\text{(d3)}\quad X_{0,0}^3+X_{0,6}^3+X_{0,12}^3-X_{6,0}^3-X_{6,6}^3-X_{6,12}^3 = 3\,\frac{\theta^{\rho_2}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_2}\left(\begin{smallmatrix}0&0\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{0,6}X_{0,12}-X_{6,0}X_{6,6}X_{6,12}),$$

$$\text{(d4)}\quad X_{0,0}^3+X_{0,6}^3+X_{0,12}^3-X_{12,0}^3-X_{12,6}^3-X_{12,12}^3 = 3\,\frac{\theta^{\rho_2}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_2}\left(\begin{smallmatrix}0&0\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{0,6}X_{0,12}-X_{12,0}X_{12,6}X_{12,12}),$$

$$\text{(d5)}\quad X_{0,0}^3+X_{6,6}^3+X_{12,12}^3-X_{0,6}^3-X_{6,12}^3-X_{12,0}^3 = 3\,\frac{\theta^{\rho_3}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_3}\left(\begin{smallmatrix}0&6\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{6,6}X_{12,12}-X_{0,6}X_{6,12}X_{12,0}),$$

$$\text{(d6)}\quad X_{0,0}^3+X_{6,6}^3+X_{12,12}^3-X_{0,12}^3-X_{6,0}^3-X_{12,6}^3 = 3\,\frac{\theta^{\rho_3}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_3}\left(\begin{smallmatrix}0&6\\0&6\end{smallmatrix}\right)}(X_{0,0}X_{6,6}X_{12,12}-X_{0,12}X_{6,0}X_{12,6}),$$

$$\text{(d7)}\quad X_{0,0}^3+X_{6,12}^3+X_{12,6}^3-X_{6,0}^3-X_{12,12}^3-X_{0,6}^3 = 3\,\frac{\theta^{\rho_4}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_4}\left(\begin{smallmatrix}0&6\\0&12\end{smallmatrix}\right)}(X_{0,0}X_{6,12}X_{12,6}-X_{6,0}X_{12,12}X_{0,6}),$$

$$\text{(d8)}\quad X_{0,0}^3+X_{6,12}^3+X_{12,6}^3-X_{12,0}^3-X_{0,12}^3-X_{6,6}^3 = 3\,\frac{\theta^{\rho_4}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)}{\theta^{\rho_4}\left(\begin{smallmatrix}0&6\\0&12\end{smallmatrix}\right)}(X_{0,0}X_{6,12}X_{12,6}-X_{12,0}X_{0,12}X_{6,6}),$$

$$(\mathrm{e}1_\rho)\qquad \sum_{a,b\in W_3}\rho\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)X_{2a,6+2b}^2X_{2a,12+2b} = \frac{\theta^\rho\left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right)}{\theta^\rho\left(\begin{smallmatrix}0&6\\4&0\end{smallmatrix}\right)}(\sum_{a,b\in W_3}\rho\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)X_{6+2a,6+2b}X_{12+2a,6+2b}X_{2a,12+2b}),$$

$$(e2_\rho)\qquad \sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X^2_{6+2a,2b}X_{12+2a,2b} = \frac{\theta^\rho\left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right)}{\theta^\rho\left(\begin{smallmatrix}4&0\\0&6\end{smallmatrix}\right)}\Big(\sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X_{6+2a,6+2b}X_{6+2a,12+2b}X_{12+2a,2b}\Big),$$

$$(e3_\rho)\qquad \sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X^2_{2a,6+2b}X_{2a,2b} = \frac{\theta^\rho\left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right)}{\theta^\rho\left(\begin{smallmatrix}0&6\\2&0\end{smallmatrix}\right)}\Big(\sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X_{6+2a,6+2b}X_{12+2a,6+2b}X_{2a,2b}\Big),$$

$$(e4_\rho)\qquad \sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X^2_{6+2a,2b}X_{2a,2b} = \frac{\theta^\rho\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right)}{\theta^\rho\left(\begin{smallmatrix}2&0\\0&6\end{smallmatrix}\right)}\Big(\sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X_{6+2a,6+2b}X_{6+2a,12+2b}X_{2a,2b}\Big),$$

$$(e5_\rho)\qquad \sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X^2_{6+2a,6+2b}X_{12+2a,12+2b} = \frac{\theta^\rho\left(\begin{smallmatrix}4&0\\4&0\end{smallmatrix}\right)}{\theta^\rho\left(\begin{smallmatrix}4&0\\4&6\end{smallmatrix}\right)}\Big(\sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X_{6+2a,12+2b}X_{6+2a,2b}X_{12+2a,12+2b}\Big),$$

$$(e6_\rho)\qquad \sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X^2_{6+2a,6+2b}X_{12+2a,2b} = \frac{\theta^\rho\left(\begin{smallmatrix}4&0\\2&0\end{smallmatrix}\right)}{\theta^\rho\left(\begin{smallmatrix}4&0\\2&6\end{smallmatrix}\right)}\Big(\sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X_{6+2a,12+2b}X_{6+2a,2b}X_{12+2a,2b}\Big),$$

$$(e7_\rho)\qquad \sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X^2_{6+2a,6+2b}X_{2a,12+2b} = \frac{\theta^\rho\left(\begin{smallmatrix}2&0\\4&0\end{smallmatrix}\right)}{\theta^\rho\left(\begin{smallmatrix}2&6\\4&0\end{smallmatrix}\right)}\Big(\sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X_{12+2a,6+2b}X_{2a,6+2b}X_{2a,12+2b}\Big),$$

$$(e8_\rho)\qquad \sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X^2_{6+2a,6+2b}X_{2a,2b} = \frac{\theta^\rho\left(\begin{smallmatrix}2&0\\2&0\end{smallmatrix}\right)}{\theta^\rho\left(\begin{smallmatrix}2&6\\2&0\end{smallmatrix}\right)}\Big(\sum_{a,b\in W_3} \rho\left(\begin{smallmatrix}a\\ b\end{smallmatrix}\right) X_{6+2a,12+2b}X_{6+2a,6+2b}X_{2a,2b}\Big).$$

In the last 8 *equations* $(e1_\rho), \dots, (e8_\rho)$, ρ *runs all the characters in* $\widehat{Z}_6^+$.

For the coefficients, we have the following:

Theorem 5. *For each* $\theta^\rho\left(\begin{smallmatrix}2x&6y\\2z&6w\end{smallmatrix}\right)$, *there is a character* χ *on* $\Gamma^2(3)$ *such that* $\chi^3 \equiv 1$, *and* $\theta^\rho\left(\begin{smallmatrix}2x&6y\\2z&6w\end{smallmatrix}\right) \in M_1(\Gamma^2(3),\chi)$. *These characters are trivial on* $\Gamma^2(9)$ *and depend only on* x, z *and* ρ. *In particular, all the coefficients of the defining equations in Theorem* 4 *are* $\Gamma^2(3)$*-invariant meromorphic functions.*

And we can show the following relations.

$$\theta^{\rho_1}\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\theta^{\rho_1}\left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right)^2 = t_1t_2^2 - t_3t_4t_5,$$

$$\theta^{\rho_1}\left(\begin{smallmatrix}0&6\\0&0\end{smallmatrix}\right)^3 = \frac{1}{24}(t_1^3 + 20t_2^3 - 4t_3^3 - 4t_4^3 - 4t_5^3 - f_1),$$

$$\theta^{1}\left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right)\theta^{1}\left(\begin{smallmatrix}0&6\\4&0\end{smallmatrix}\right)^2 = \frac{1}{9}(t_3t_4t_5 + t_1t_2t_4 + t_1t_2t_5 + t_1t_4t_5 + t_2^2t_3 + t_2t_3t_4 + t_3t_4^2 + t_3t_5^2),$$

$$\begin{aligned}\theta^{\rho_1}\left(\begin{smallmatrix}0&6\\4&0\end{smallmatrix}\right)^3 = \frac{1}{216}(&-t_1^3 + 4(t_2^3 + t_3^3 + t_4^3 + t_5^3) + f_1 + 6t_1^2t_3\\ &+ 2f_3 + 24(t_2t_4t_5 + t_2^2t_4 + t_2t_4^2 + t_2^2t_5 + t_2t_5^2 + t_4t_5^2 + t_4^2t_5)).\end{aligned}$$

To prove the theorem, we can show that the group $\Gamma^2(3)/\Gamma^2(36)$ are generated by the following elements:

$$\begin{pmatrix}1_2 & 3S\\ 0 & 1_2\end{pmatrix},\quad \begin{pmatrix}1_2 & 0\\ 3S & 1_2\end{pmatrix},\quad {}^tS = S,$$

$$\begin{pmatrix}U_i & 0\\ 0 & {}^tU_i^{-1}\end{pmatrix}\ (1 \le i \le 3),\quad U_1 = \begin{pmatrix}1&1\\0&1\end{pmatrix},\ U_2 = \begin{pmatrix}1&0\\1&1\end{pmatrix},\ U_3 = \begin{pmatrix}4&3\\-3&-2\end{pmatrix},$$

$$\begin{pmatrix}1_2 & \begin{smallmatrix}1&0\\0&0\end{smallmatrix}\\ 0 & 1_2\end{pmatrix}\begin{pmatrix}1_2 & 0\\ \begin{smallmatrix}3&0\\0&0\end{smallmatrix} & 1_2\end{pmatrix}\begin{pmatrix}1_2 & \begin{smallmatrix}-1&0\\0&0\end{smallmatrix}\\ 0 & 1_2\end{pmatrix}.$$

By the theory of the theta series of quadratic forms (cf. [A, Chapter 1, 2]), we can check the modularity for the above generators directory.

4. Explicit form of the defining equations

Finally we consider the problem: find the relations derived from quadratic relations among the cubic relations. Since $\dim(\ker\varphi_2 \otimes H^0(A,L)) = 9\times 9 = 81 < 84 = \dim\ker\varphi_3$, we need at least 3 cubic relations. In fact we have the following theorem.

Theorem 6 (Main Theorem). *Let X_{00}, X_{01}, X_{02}, X_{10}, X_{11}, X_{12}, X_{20}, X_{21} and X_{22} be the coordinate of $\mathbb{P}^8$. The defining equations of an abelian surface $\mathbb{C}^2/(\tau\mathbb{Z}^2+\mathbb{Z}^2)$ is given by the following* 12 *equations.*

$$\begin{aligned}
&h_1X_{00}^2 + h_2X_{10}X_{20} + h_3X_{01}X_{02} + h_4X_{11}X_{22} + h_5X_{12}X_{21} = 0,\\
&h_1X_{10}^2 + h_2X_{20}X_{00} + h_3X_{11}X_{12} + h_4X_{21}X_{02} + h_5X_{22}X_{01} = 0,\\
&h_1X_{20}^2 + h_2X_{00}X_{10} + h_3X_{21}X_{22} + h_4X_{01}X_{12} + h_5X_{02}X_{11} = 0,\\
&h_1X_{01}^2 + h_2X_{11}X_{21} + h_3X_{02}X_{00} + h_4X_{12}X_{20} + h_5X_{10}X_{22} = 0,\\
&h_1X_{11}^2 + h_2X_{21}X_{01} + h_3X_{12}X_{10} + h_4X_{22}X_{00} + h_5X_{20}X_{02} = 0,\\
&h_1X_{21}^2 + h_2X_{01}X_{11} + h_3X_{22}X_{20} + h_4X_{02}X_{10} + h_5X_{00}X_{12} = 0,\\
&h_1X_{02}^2 + h_2X_{12}X_{22} + h_3X_{00}X_{01} + h_4X_{10}X_{21} + h_5X_{11}X_{20} = 0,\\
&h_1X_{12}^2 + h_2X_{22}X_{02} + h_3X_{10}X_{11} + h_4X_{20}X_{01} + h_5X_{21}X_{00} = 0,\\
&h_1X_{22}^2 + h_2X_{02}X_{12} + h_3X_{20}X_{21} + h_4X_{00}X_{11} + h_5X_{01}X_{10} = 0,
\end{aligned}$$

$$\begin{aligned}
&X_{00}^3 + X_{01}^3 + X_{02}^3 + X_{10}^3 + X_{11}^3 + X_{12}^3 + X_{20}^3 + X_{21}^3 + X_{22}^3\\
&\quad = 3\,\frac{t_1}{t_2}(X_{00}X_{10}X_{20} + X_{01}X_{11}X_{21} + X_{02}X_{12}X_{22}),\\
&\quad = 3\,\frac{t_1}{t_3}(X_{00}X_{01}X_{02} + X_{10}X_{11}X_{12} + X_{20}X_{21}X_{22}),\\
&\quad = 3\,\frac{t_1}{t_4}(X_{00}X_{11}X_{22} + X_{01}X_{12}X_{20} + X_{02}X_{21}X_{10}).
\end{aligned}$$

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